Soluție

a) $f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}, \forall x \in (0, +\infty)$

b) $y - f(e) = f'(e)(x - e)$

$y - \left(\frac{1}{e} + 1\right) = \frac{e-1}{e^2}(x - e)$

$e^2 y + x(1-e) - 2e = 0$

c) $f(e) + f\left(\frac{1}{e}\right) > 2 \Leftrightarrow \frac{1}{e} + 1 + e - 1 > 2$

$\Leftrightarrow \frac{1}{e} + e > 2 \Leftrightarrow 1 + e^2 - 2e > 0 \Leftrightarrow (1-e)^2 > 0$

d) $f'(x) = 0 \Rightarrow x = 1$

$x \in (0,1] \Rightarrow f'(x) \leq 0 \Rightarrow f$ descrescătoare

$x \in [1, +\infty) \Rightarrow f'(x) \geq 0 \Rightarrow f$ crescătoare

e) $\lim\limits_{x \to \infty} \left[x\left(f(x) - \ln x \right) \right] = \lim\limits_{x \to \infty} \left[x\left(\frac{1}{x} + \ln x - \ln x \right) \right] = 1$

f) $l_s(1) = \lim\limits_{x \nearrow 1} (3\alpha x + 11) = 3\alpha + 11$

$l_d(1) = \lim\limits_{x \searrow 1} \left(\alpha^2 + 1\right) = \alpha^2 + 1$

$g(1) = \alpha^2 + 1$

g continuă $x = 1 \Rightarrow l_s(1) = l_d(1) = g(1)$

$\Rightarrow \alpha^2 + 1 = 3\alpha + 11 \Rightarrow \alpha^2 - 3\alpha - 10 = 0 \Rightarrow \alpha_1 = 5, \alpha_2 = -2$