Soluție

1. a) $f'(x)=\left(\dfrac{x^2-1}{x^2+1}\right)'=\dfrac{\left(x^2-1\right)'\left(x^2+1\right)-\left(x^2+1\right)'\left(x^2-1\right)}{\left(x^2+1\right)^2}=\dfrac{4x}{\left(x^2+1\right)^2},\quad \forall x\in\mathbb{R}.$

b) Avem din punctul a) $f'(x)=\dfrac{4x}{\left(x^2+1\right)^2}, \forall x\in\mathbb{R}$. $f'(x)=0\Rightarrow x=0$. Din tabelul de variație al funcției

obținem $\begin{cases} f \searrow & \textit{pentru } x\in(-\infty,0] \\ f \nearrow & \textit{pentru } x\in[0,+\infty) \end{cases}$.

c) Din ipoteză $g(x)=f(x)+f\left(\dfrac{1}{x}\right)=\dfrac{x^2-1}{x^2+1}+\dfrac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1}=0, \forall x\in\mathbb{R}^*.$

Deci $\lim\limits_{x\to0}\dfrac{g(x)+g\left(x^2\right)+g\left(x^3\right)+\ldots+g\left(x^{2008}\right)+x^{2010}}{x^{2009}}=\lim\limits_{x\to0}\dfrac{\overbrace{0+0+\ldots+0}^{de\,2008\,ori}+x^{2010}}{x^{2009}}=\lim\limits_{x\to0}x=0.$

2. a) $I_0=\int_e^{e^2}x\ln^0 x\,dx=\int_e^{e^2}x\,dx=\left.\dfrac{x^2}{2}\right|_e^{e^2}=\dfrac{e^4-e^2}{2}.$

b) $x\in\left[e,e^2\right]\Rightarrow 1\le\ln x\le2\Rightarrow x\cdot\ln^n x\le x\cdot\ln^{n+1}x,\ \forall x\in\left[e,e^2\right]$ și $\forall n\in\mathbb{N}$. Integrând obținem $I_n\le I_{n+1}$.

c) $I_n=\int_e^{e^2}x\cdot\ln^n x\,dx=\int_e^{e^2}\left(\dfrac{x^2}{2}\right)'\cdot\ln^n x\,dx=\left.\dfrac{x^2}{2}\cdot\ln^n x\right|_e^{e^2}-\int_e^{e^2}\dfrac{x^2}{2}\cdot\left(\ln^n x\right)'dx=\dfrac{e^4\cdot2^n}{2}-\dfrac{e^2}{2}-\dfrac{n}{2}\int_e^{e^2}x\cdot\ln^{n-1}x\,dx=$

$=\dfrac{e^2\left(e^2\cdot2^n-1\right)}{2}-\dfrac{n}{2}I_{n-1}$, oricare ar fi $n\in\mathbb{N}^*$.