Soluție

1.a $\lim\limits_{x\to\infty} f(x) = 1$

y=1 asimptota orizontala spre $+\infty$

b $f'(x) = \dfrac{2x+1}{2\sqrt{x^2+x+1}} - \dfrac{2x-1}{2\sqrt{x^2-x+1}}$

$f'(x) \neq 0,\ \mathrm{f}'(0) > 0 \ \Rightarrow\ \mathrm{f}'(\mathrm{x}) > 0,\ \forall \mathrm{x} \in \mathbb{R}$

f strict crescatoare pe $\mathbb{R}$

c $\sum\limits_{k=1}^{n} f(k) = \sum\limits_{k=1}^{n}\left(\sqrt{k(k+1)+1} - \sqrt{k(k-1)+1}\right) = \sqrt{n^2+n+1} - 1$

$\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{n^2+n+1}-1}{n}\right)^n = e^{\lim\limits_{n\to\infty}\left(\sqrt{n^2+n+1}-n-1\right)} = e^{-\frac{1}{2}}$

2.a $I_1 = -\dfrac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\,dx = -\dfrac{1}{3}\left(1-x^2\right)\sqrt{1-x^2}\ \Big|_0^1 = \dfrac{1}{3}$

b $I_n = \lim\limits_{\substack{a\to 1\\ a<1}} J_{a,n},\ \text{unde } \mathrm{J_a} = \int_0^a \mathrm{x^n}(\sqrt{1-x^2})dx$

$J_{a,n} = \int_0^a \dfrac{x^n - x^{n+2}}{\sqrt{1-x^2}} = \int_0^a x^{n+1}(\sqrt{1-x^2})dx - \int_0^a x^{n-1}(\sqrt{1-x^2})dx$

$= \mathrm{a}^{\mathrm{n}+1}\sqrt{1-a^2} - a^{n-1}\sqrt{1-a^2} - (n+1)J_{a,n+1} + (n-1)J_{a,n-2}$

Trecând la limita obținem relația dorită

c $0 \le x^n\sqrt{1-x^2} \le x^n,\ \forall x \in [0;1]$

$0 \le I_n \le \int_0^1 x^n dx = \dfrac{1}{n+1}$

$\Rightarrow \lim\limits_{n\to\infty} I_n = 0$