Soluție

1.a) $f'(x)=\left(\dfrac{x-e^x}{x+e^x}\right)'=\dfrac{\left(1-e^x\right)\left(x+e^x\right)-\left(1+e^x\right)\left(x-e^x\right)}{\left(x+e^x\right)^2}=\dfrac{2e^x(1-x)}{\left(x+e^x\right)^2}$.

b) Avem $\lim\limits_{x\to+\infty}\dfrac{x}{e^x}=0$ (demonstrație cu L'Hôspital, sau folosind $x\ll e^x$, pentru $x\to+\infty$ **).** $\lim\limits_{x\to+\infty}f(x)=$

$=\lim\limits_{x\to+\infty}\dfrac{x-e^x}{x+e^x}=\lim\limits_{x\to+\infty}\dfrac{e^x\left(\dfrac{x}{e^x}-1\right)}{e^x\left(\dfrac{x}{e^x}+1\right)}\overset{\lim\limits_{x\to+\infty}\frac{x}{e^x}=0}{=}-1\Rightarrow d:y=-1$ este asimptotă orizontală la G_f spre $+\infty$.

c) $f'(x)=\dfrac{2e^x(1-x)}{\left(x+e^x\right)^2}=0\Rightarrow x=1\in[0,+\infty)$. Din tabelul de variație al funcției f deducem că $A\left(1,\dfrac{1-e}{1+e}\right)$ este punct de maxim. Cum $f(1)=\dfrac{1-e}{1+e}$ și $f(0)=\lim\limits_{x\to+\infty}f(x)\overset{cf.b)}{=}-1$, obținem $-1\le f(x)\le\dfrac{1-e}{1+e}$, $\forall x\ge0$.

2.a) $I_1=\int_0^1\dfrac{x}{x+1}dx=\int_0^1\dfrac{(x+1)-1}{x+1}dx=\int_0^1\left(1-\dfrac{1}{x+1}\right)dx\underset{u'(x)=1}{\overset{u(x)=x+1}{=}}\int_0^1\left(1-\dfrac{u'(x)}{u(x)}\right)dx=\left(x-\ln\left|u(x)\right|\right)\Big|_0^1=1-\ln2$.

b) $I_{n+1}+I_n=\int_0^1\left(\dfrac{x^{n+1}}{x+1}+\dfrac{x^n}{x+1}\right)dx=\int_0^1\dfrac{x^n(x+1)}{x+1}dx=\int_0^1x^n\,dx=\dfrac{x^{n+1}}{n+1}\Big|_0^1=\dfrac{1}{n+1}$.

c) Integrând $\Rightarrow\int_0^1\dfrac{x^n}{2}dx\le\int_0^1\dfrac{x^n}{x+1}dx\le\int_0^1x^n\,dx\Rightarrow\dfrac{x^{n+1}}{2(n+1)}\Big|_0^1\le I_n\le\dfrac{x^{n+1}}{n+1}\Big|_0^1\Rightarrow\dfrac{1}{2(n+1)}\le I_n\le\dfrac{1}{n+1}\overset{n=2008}{\Rightarrow}$ q.e.d.