Soluție

a) $(x-1)\cdot f(x)=(x-1)\cdot\dfrac{x^3+x^2+x+1}{x^3}=\dfrac{x^4+x^3+x^2+x-x^3-x^2-x-1}{x^3}=\dfrac{x^4-1}{x^3}=x-\dfrac{1}{x^3}=g(x)$

b) Fie $F:\mathbb{R}\to\mathbb{R}$ primitivă a funcției $f:\mathbb{R}\to\mathbb{R}$; derivând $F(x)=\int f(x)dx \Leftrightarrow F'(x)=f(x)$

Deoarece $x\in(0;+\infty)\Rightarrow x^2,x^3>0\Rightarrow\quad f(x)=1+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}>0,\forall x\in(0;+\infty)\Rightarrow$

$F'(x)=f(x)>0, x\in(0,+\infty)\Rightarrow F\uparrow, x\in(0,+\infty)$ (F este strict crescătoare pe $(0,+\infty)$)

c) $F(x)=\int f(x)dx=\int\left(1+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}\right)dx=x+\ln|x|-\dfrac{1}{x}-\dfrac{1}{2x^2}+\ell; A\left(1,-\dfrac{1}{2}\right)\in G_{F(x)}\Rightarrow F(1)=-\dfrac{1}{2}\Rightarrow$

$\ell=0\Rightarrow F(x)=x+\ln x-\dfrac{1}{x}-\dfrac{1}{2x^2}, x>0$

d) $\int_1^2 x^3 f(x)dx=\int_1^2\left(x^3+x^2+x+1\right)dx=\left.\left(\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}+x\right)\right|_1^2=\dfrac{103}{12}$

e) notăm $g:[a,b]\to\mathbb{R}, g(x)=x-\dfrac{1}{x^3}$ și folosim proprietatea: $0\le g(x),\forall x\in[a,b]\Rightarrow 0\le\int_a^b g(x)dx$

$x-\dfrac{1}{x^3}=\dfrac{x^4-1}{x^3}\ge 0,\forall x\ge 1\Rightarrow g(x)\ge 0,\forall x\ge 1\Rightarrow$ prin înlocuire, avem: $\int_a^b g(x)dx\ge 0,$ oricare ar fi $a,b\in\mathbb{R}, 1\le a\le b$

f) $\int_1^e g\left(\dfrac{1}{x}\right)dx=\int_1^e\left(\dfrac{1}{x}-x^3\right)dx=\left.\left(\ln|x|-\dfrac{x^4}{4}\right)\right|_1^e=\dfrac{5-e^4}{4}$