Soluție

1. a) $f(x)=\frac{1}{x}-\frac{1}{x+1} \Leftrightarrow f(x)=\frac{x+1-x}{x(x+1)} \Leftrightarrow f(x)=\frac{1}{x(x+1)}$, adevărat, pentru orice $x>0$.

b) $f'(x)=\left(\frac{1}{x(x+1)}\right)'=-\frac{\left(x(x+1)\right)'}{\left(x(x+1)\right)^2}=-\frac{2x+1}{x^2(x+1)^2}=\frac{x^2-(x+1)^2}{x^2(x+1)^2}=\frac{1}{(x+1)^2}-\frac{1}{x^2},\ \forall x>0$.

c) $\lim\limits_{x\to+\infty} x f(x) f\left(\frac{1}{x}\right)=\lim\limits_{x\to+\infty}\left(x\cdot\frac{1}{x(x+1)}\cdot\frac{1}{\frac{1}{x}\left(\frac{1}{x}+1\right)}\right)=\lim\limits_{x\to+\infty}\frac{x^2}{(x+1)^2}\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}\lim\limits_{x\to+\infty}\frac{2x}{2(x+1)}\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}\lim\limits_{x\to+\infty}\frac{2}{2}=1$.

2. a) $I_0+I_2=\int_1^{\sqrt{3}}\left(\frac{1}{x^2+1}+\frac{1}{x^2\left(x^2+1\right)}\right)dx=\int_1^{\sqrt{3}}\frac{x^2+1}{x^2\left(x^2+1\right)}dx=\int_1^{\sqrt{3}}\frac{1}{x^2}dx=-\frac{1}{x}\Big|_1^{\sqrt{3}}=\frac{\sqrt{3}-1}{\sqrt{3}}$

b) $I_1=\int_1^{\sqrt{3}}\frac{1}{x\left(x^2+1\right)}dx=\int_1^{\sqrt{3}}\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx \overset{u(x)=x^2+1}{\underset{u'(x)=2x}{=}} \int_1^{\sqrt{3}}\frac{1}{x}dx-\frac{1}{2}\int_1^{\sqrt{3}}\frac{u'(x)}{u(x)}dx=$

$=\ln|x|\Big|_1^{\sqrt{3}}-\frac{1}{2}\ln|u(x)|\Big|_1^{\sqrt{3}}=\ln\sqrt{3}-\frac{1}{2}(\ln 4-\ln 2)=\frac{1}{2}\ln\frac{3}{2}$.

c) $I_n+I_{n-2}=\int_1^{\sqrt{3}}\left(\frac{1}{x^n\left(x^2+1\right)}+\frac{1}{x^{n-2}\left(x^2+1\right)}\right)dx=\int_1^{\sqrt{3}}\frac{1}{x^n}dx=\frac{x^{-n+1}}{-n+1}\Big|_1^{\sqrt{3}}=\frac{1}{n-1}\left(1-\frac{1}{\left(\sqrt{3}\right)^{n-1}}\right),\ \forall n\in\mathbb{N}, n\geq 2.$